A Theorem on Frequency Function for Multiple-Valued Dirichlet Minimizing Functions *

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February 2, 2008

Abstract

This paper discusses the frequency function of multiple-valued Dirichlet minimizing functions in the special case when the domain and range are both two dimensional. It shows that the frequency function must be of value $k/2$ for some nonnegative integer k. Futhermore, by looking at the blowing-up functions, we characterize the local behavior of the original Dirichlet minimizing function.

2000 Mathematics Subject Classification: Primary 49Q20

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1 Introduction

Frequency function for multiple-valued functions was introduced by Almgren in [AF] to study the branching behavior for multiple-valued Dirichlet minimizing functions:

$$N(r) = \frac{r\displaystyle\int_{\mathbb{B}_r^m(a)} |Df|^2}{\displaystyle\int_{\partial\mathbb{B}_r^m(a)} |f|^2}.$$

For Dirichlet minimizing functions, $N(r)$ is nondecreasing in r. Almgren establishes this monotonicity by certain range and domain deformations, called "squashing" and "squeezing". The monotonicity property enables one to prove by dimension reduction that such multiple-valued functions have branched sets of codimension at least two.
To get a better idea about this frequency function, consider a harmonic function on $\mathbb{R}^2$, and express it in terms of polar coordinates: $u(r,\theta)$. If we fix r, we can expand the resulting function of θ as a Fourier series. Now as r decreases, the higher frequency terms in the Fourier series die off faster than the lower frequency terms.
Monotonicity of frequency functions have been used in some other work, see [GL], [LF], [GS].
The motivation of this paper was trying to characterize multiple-valued Dirichlet minimizing functions $f : \mathbb{R}^2 \to \mathbb{Q}_2(\mathbb{R}^2)$, which is homogeneous of some positive degree. There are a lot of them like $z^{1/2}, \pm z, z^{3/2}$. In general, any function of the form

$$z^N, \text{ for some positive real number } N$$

could be a candidate. One thing worth mentioning is that the frequency function at the origin of z^N is exactly N. However, not every N gives a 2-valued function because the function has to match up itself once going around the circle one time. For example, consider the function

$$f : (r,\theta) \to [[(r^N\cos(N\theta), r^N\sin(N\theta))]] + [[(-r^N\cos(N\theta), -r^N\sin(N\theta))]],$$

when $N = 1/4$.

$$f(r,0) = [[(r^N,0)]] + [[(-r^N,0)]], f(r,2\pi) = [[(0,r^N)]] + [[(0,-r^N)]].$$

They do not match. We will see that in this case, only by choosing $N = k/2$ for some positive integer k makes f a well-defined 2-valued function. This is basically the main ingredient of the proof of our main theorem, matching up values for $\theta = 0$ and $\theta = 2\pi$. More precisely, given a multiple-valued Dirichlet minimizing function $f : \mathbb{R}^2 \to \mathbb{Q}_2(\mathbb{R}^2)$, with $\mathcal{N}(0) = N$, we use the blowing-up analysis to get a Dirichlet minimizing function $g : \mathbb{R}^2 \to \mathbb{Q}_2(\mathbb{R}^2)$ of homogeneous degree N with the same frequency N at the origin. By doing the matching up business for g, we succeed in proving $N = k/2$ for some nonnegative integer k. A by-product of this proof is the characterization of local behavior of original

function f near the origin.
So a natural question is whether we have similar results in higher dimension, either domain or the range, or higher multiplicity Q. For functions $f : \mathbb{R}^2 \to \mathbb{Q}_3(\mathbb{R}^2)$, it would take a lot more work to matching up values. Therefore some other easier methods are expected to give a full answer to this question in general.

2 Preliminaries

We refer to [AF], [ZW1] for most of notations, definitions and known results about multiple-valued functions. For reader's convenience, here we state some useful results. The proofs of them can be found in [AF].

Theorem 2.1 ([**AF**], §2.6). ***Hypotheses:***
(a) $0 < r_0 < \infty$.
(b) $A \subset \mathbb{R}^m$ *is connected, open, and bounded with* $\mathbb{U}^m_{r_0}(0) \subset A$. ∂A *is an* $m-1$ *dimensional submanifold of* $\mathbb{R}^m$ *of class 1.*
(c) $f : A \to \mathbb{Q}$ *is strictly defined and is Dir minimizing.*
(d) $D, H, N : (0, r_0) \to \mathbb{R}$ *are defined for* $0 < r < r_0$ *by setting*

$$\begin{aligned} D(r) &= Dir(f; \mathbb{B}^m_r(0)) \\ H(r) &= \int_{\partial \mathbb{B}^m_r(0)} \mathcal{G}(f(x), Q[[0]])^2 d\mathcal{H}^{m-1}x \\ N(r) &= rD(r)/H(r) \text{ provided } H(r) > 0. \end{aligned}$$

(e) $\mathcal{N} : A \to \mathbb{R}$ *is defined for* $x \in A$ *by setting*

$$\mathcal{N}(x) = \lim_{r \downarrow 0} rDir(f; \mathbb{B}^m_r(x)) / \int_{\partial \mathbb{B}^m_r(x)} \mathcal{G}(f(z), Q[[0]])^2 d\mathcal{H}^{m-1}z$$

provided this limit exists.
(f) $H(r) > 0$ *for some* $0 < r < r_0$.
Conclusions.
(1) $\eta \circ f \in \mathcal{Y}_2(A, \mathbb{R}^n)$ *is Dir minimizing and harmonic.*
(2) $N(r)$ *is defined for each* $0 < r < r_0$ *and is nondecreasing.*
(3) $\mathcal{N}(0) = \lim_{r \downarrow 0} N(r)$ *exists.*
(4)
$A = A \cap \{x : \textit{for some } 0 < r < dist(x, \partial A), \int_{\partial \mathbb{U}^m_r(x)} \mathcal{G}(f(z), Q[[0]])^2 d\mathcal{H}^{m-1} > 0\}$.
(5) $\mathcal{N}(x)$ *is well defined for each* $x \in A$ *and is upper semi-continuous as a function of* x.
(6) In case $N(r) = \mathcal{N}(0)$ *for* $\mathcal{L}^1$ *almost all* $0 < r < r_0$, *then*

$$f(x) = \mu[(r/r_1)^{\mathcal{N}(0)}]_\sharp f(r_1 x/|x|)$$

for $\mathcal{L}^{m-1}$ *almost all* $x \in \partial \mathbb{U}^m_r(0)$ *and each* $0 < r_1 < r_0$.

Theorem 2.2 ([AF], §2.13). ***Hypotheses.***
(a) In case $m = 2$, $\omega_{2.13} = 1/Q$.
(b) In case $m \geq 3, 0 < \epsilon_Q < 1$ *is as defined as in [AF], §2.11 and* $0 < \omega_{2.13} < 1$ *is defined by the requirement*

$$m - 2 + 2\omega_{2.13} = (m-2)(1+\epsilon_Q)/(1-\epsilon_Q).$$

(c)

$$\Gamma_{2.13} = 4^{1-\omega_{2.13}}[2^{m/2}/(1-2^{-\omega_{2.13}}) + 3\cdot 2^{m-1+\omega_{2.13}}](m\alpha(m))^{-1/2} Lip(\xi)^2 Lip(\xi^{-1}).$$

(d) $f \in \mathcal{Y}_2(\mathbb{R}^m, \mathbb{Q})$ *is strictly defined and* $f|\mathbb{U}_1^m(0)$ *is Dir minimizing with* $Dir(f; \mathbb{B}_1^m(0)) > 0$.
(e) $\mathcal{N}(0) = \lim_{r\downarrow 0} r Dir(f; \mathbb{B}_r^m(0)) / \int_{x\in\partial\mathbb{B}_r^m(0)} \mathcal{G}(f(x), Q[[0]])^2 d\mathcal{H}^{m-1}x$.
Conclusions.
(1) For each $z \in \mathbb{U}_1^m(0), 0 < r < 1 - |z|$, *and* $0 < s \leq 1$,

$$Dir(f; \mathbb{B}_{sr}^m(z)) \leq s^{m-2+2\omega_{2.13}} Dir(f; \mathbb{B}_r^m(z)).$$

(2) Whenever $0 < \delta < 1$ *and* $p, q \in \mathbb{B}_{1-\delta}^m(0)$,

$$\mathcal{G}(f(p), f(q)) \leq \Gamma_{2.13}\delta^{-m/2} Dir(f; \mathbb{B}_1^m(0))^{1/2}|p-q|^{\omega_{2.13}},$$

in particular, $f|\mathbb{B}_{1-\delta}^m(0)$ *is Hölder continuous with exponent* $\omega_{2.13}$.
(3) Either $f(0) = Q[[0]]$ *and* $\mathcal{N}(0) \geq \omega_{2.13}$ *or* $f(0) \neq Q[[0]]$ *and* $\mathcal{N}(0) = 0$.
(4) Suppose $f(0) = Q[[0]]$ *and* $1/2 > r(1) > r(2) > r(3) > \cdots > 0$ *with* $0 = \lim_{i\to\infty} r(i)$. *Then there is a subsequence* $i_1, i_2, i_3, \cdots$ *of* $1, 2, 3, \cdots$ *and a function* $g : \mathbb{B}_1^m(0) \to \mathbb{Q}$ *with the following properties:*
(a) g *is the uniform limit as* $k \to \infty$ *of the functions*

$$\mu(Dir(f \circ \mu[r(i_k)]; \mathbb{B}_1^m(0))^{-1/2})_\sharp \circ f \circ \mu[r(i_k)]|\mathbb{B}_1^m(0).$$

(b) $g|\mathbb{U}_1^m(0) \in \mathcal{Y}_2(\mathbb{U}_1^m(0), \mathbb{Q})$ *is Dir minimizing with* $Dir(g; \mathbb{B}_1^m(0)) = 1$.
(c) $\int_{x\in\partial\mathbb{B}_1^m(0)} \mathcal{G}(g(x), Q[[0]])^2 d\mathcal{H}^{m-1}x = 1/\mathcal{N}(0)$.
(d) $g(0) = Q[[0]]$ *and for each* $x \in \mathbb{B}_1^m(0) \sim \{0\}$,

$$g(x) = \mu(|x|^{\mathcal{N}(0)})_\sharp \circ g(x/|x|).$$

(e) For each $p, q \in \mathbb{B}_1^m(0)$,

$$\mathcal{G}(g(p), g(q)) \leq 2^{m/2-\omega_{2.13}+\mathcal{N}(0)}\Gamma_{2.13}|p-q|^{\omega_{2.13}}.$$

(5) Corresponding to each bounded open set A *such that* ∂A *is a compact* $m-1$ *dimensional submanifold of* $\mathbb{R}^m$ *of class 1, there is a constant* $0 < \Gamma_A < \infty$ *with the following property. Whenever* $g \in \mathcal{Y}_2(A, \mathbb{Q})$ *is Dir minimizing and* $p, q \in A$,

$$\mathcal{G}(g(p), g(q)) \leq \Gamma_A Dir(g; A)^{1/2} \sup\{dist(p, \partial A)^{-m/2}, dist(q, \partial A)^{-m/2}\}|p-q|^{\omega_{2.13}}.$$

Theorem 2.3 ([**AF**], §2.14)**.** *(1) Let* $\mu \in \{1, 2, \cdots, Q\}$ *and suppose* $f_1, f_2, \cdots, f_Q \in \mathcal{Y}_2(\mathbb{U}_1^\mu(0), \mathbb{R}^n)$ *are strictly defined. Then* $f = \sum_{i=1}^{Q} [[f_i]] \in \mathcal{Y}_2(\mathbb{U}_1^\mu, \mathbb{Q})$*. Furthermore, in case* f *is Dir minimizing, so is each* $f_i, i = 1, 2, \cdots, Q$*.*
(2)Suppose $f \in \mathcal{Y}_2(\mathbb{U}_1^m(0), \mathbb{Q})$ *is strictly defined and Dir minimizing. Then the function*

$$\sigma : \mathbb{U}_1^m(0) \to \{1, 2, \cdots, Q\},$$

$$\sigma(x) = card\,[spt(f(x))] \text{ for } x \in \mathbb{U}_1^m(0),$$

is lower semi-continuous, the set

$$\Sigma = \mathbb{U}_1^m(0) \cap \{x : \sigma \text{ is not continuous at } x\}$$

is closed in $\mathbb{U}_1^m(0)$ *with Hausdorff dimension not exceeding* $m-2$*, and the set* $\mathbb{U}_1^m(0) \sim \Sigma$ *is open and path connected. Furthermore, there exist* $J \in \{1, 2, \cdots, Q\}$ *and* $k_1, k_2, \cdots, k_J \in \{1, 2, \cdots, Q\}$ *with* $k_1 + k_2 + \cdots + k_J = Q$ *with the following properties: whenever* $W \subset \mathbb{U}_1^m(0) \sim \Sigma$ *is open and simply connected, there are harmonic functions* $f_1, f_2, \cdots, f_J : W \to \mathbb{R}^n$ *such that* $f(x) = \sum_{i=1}^{J} k_i [[f_i(x)]]$ *and* $J = card\,\{f_1(x), f_2(x), \cdots, f_J(x)\}$ *for each* $x \in W$*.*

3 Main Theorem

Theorem 3.1. ***Hypotheses.***
(a) $m = 2, n = 2, Q = 2, \omega_{2.13} = 1/2$*.*
(b) $f \in \mathcal{Y}_2(\mathbb{R}^2, \mathbb{Q}(\mathbb{R}^2))$ *is strictly defined and* $f|\mathbb{U}_1^2(0)$ *is Dir minimizing with* $Dir(f; \mathbb{B}_1^2(0)) > 0$*.*
(c) $f(0) = 2[[0]]$*,*
(d) $\mathcal{N}(0) = \lim_{r \downarrow 0} r Dir(f; \mathbb{B}_r^2(0)) / \int_{x \in \partial \mathbb{B}_r^2(0)} \mathcal{G}(f(x), 2[[0]])^2 d\mathcal{H}^1 x$
Conclusion.

$$\mathcal{N}(0) = k/2, \text{ for some positive integer } k.$$

Proof. According to Theorem 2.2, we know that $\mathcal{N}(0) \geq \omega_{2.13} > 0$ and suppose $1/2 > r(1) > r(2) > r(3) > \cdots > 0$ with $0 = \lim_{i \to \infty} r(i)$, then there is a subsequence $i_1, i_2, i_3, \cdots$ of $1, 2, 3, \cdots$ and a function $g : \mathbb{B}_1^2(0) \to \mathbb{Q}$ with the following properties:
(1) g is the uniform limit as $k \to \infty$ of the functions

$$\mu(\mathrm{Dir}(f \circ \mu[r(i_k)]; \mathbb{B}_1^2(0))^{-1/2})_\sharp \circ f \circ \mu[r(i_k)]|\mathbb{B}_1^2(0).$$

(2) $g|\mathbb{U}_1^2(0) \in \mathcal{Y}_2(\mathbb{U}_1^2(0), \mathbb{Q})$ is Dir minimizing with $\mathrm{Dir}(g; \mathbb{B}_1^2(0)) = 1$.
(3) $\int_{x \in \partial \mathbb{B}_1^2(0)} \mathcal{G}(g(x), 2[[0]])^2 d\mathcal{H}^1 x = 1/\mathcal{N}(0)$.
(4) $g(0) = 2[[0]]$ and for each $x \in \mathbb{B}_1^2(0) \sim \{0\}$,

$$g(x) = \mu(|x|^{\mathcal{N}(0)})_\sharp \circ g(x/|x|).$$

(5) For each $p, q \in \mathbb{B}_1^2(0)$,

$$\mathcal{G}(g(p), g(q)) \leq 2^{1/2+\mathcal{N}(0)} \Gamma_{2.13} |p - q|^{1/2}.$$

First of all, we claim

$$\Sigma(g) = \emptyset, \text{ or } \{0\}.$$

This comes from the fact that g is homogeneous of degree $\mathcal{N}(0)$. If σ is not continuous at some nonzero point y, σ is not continuous at every point on the ray $ty, t \in (0,1)$. Then the Hausdorff dimension of Σ is at least one, which is in contradiction to Theorem 2.3(2).
Rest of the proof is divided into two cases: $\Sigma = \emptyset$ and $\Sigma = \{0\}$.

3.1 $\Sigma = \{0\}$

If $\Sigma = \{0\}$, applying Theorem 2.3(2) to the function g, we get $J = 2$. This is because otherwise if $J = 1$, then for any point $x \in \mathbb{U}_1^2(0) \sim \{0\}$, $\sigma(x) = 1$. Therefore σ is a constant function on $\mathbb{U}_1^2(0)$, which means $\Sigma = \emptyset$, a contradiction to our assumption.
Take $W = \mathbb{U}_1^2(0) \sim \{(x,0), x \geq 0\}$ in Theorem 2.3(2), we have

$$g(x) = \sum_{i=1}^{2} [[h_i(x)]], x \in W$$

for harmonic functions $h_i : W \to \mathbb{R}^2, i = 1,2$ and $h_1(x) \neq h_2(x), \forall x \in W$.
For simplicity, we denote $\mathcal{N}(0)$ as N. Since g is homogeneous of degree N, so is $h_i, i = 1,2$. Hence we can write

$$g(r,\theta) = [[r^N g_1(\theta)]] + [[r^N g_2(\theta)]], 0 < r \leq 1, 0 < \theta < 2\pi$$

where $g_i : (0,2\pi) \to \mathbb{R}^2, i = 1,2$, and $r^N g_i$ is harmonic, $i = 1,2$.
Moreover, in spirit of Theorem 2.3(1), $r^N g_i$ must be Dir minimizing, hence conformal on W for $i = 1,2$.
Let

$$g_1(\theta) = (g_1^1(\theta), g_1^2(\theta)), g_2(\theta) = (g_2^1(\theta), g_2^2(\theta)).$$

The Laplacian operator in polar coordinate can be expressed as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Do the computation, we have

$$\frac{\partial^2}{\partial r^2}(r^N g_i^j(\theta)) = N(N-1)r^{N-2} g_i^j(\theta),$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r^N g_i^j(\theta)) = N r^{N-2} g_i^j(\theta),$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^N g_i^j(\theta)) = r^{N-2}[g_i^j(\theta)]''.$$

Therefore, $\Delta(r^N g_i^j(\theta)) = r^{N-2}[N^2 g_i^j(\theta) + [g_i^j(\theta)]''] = 0, i = 1,2, j = 1,2.$
Hence we can represent $g_i^j(\theta)$ as

$$g_i^j(\theta) = a_i^j \cos(N\theta) + b_i^j \sin(N\theta), i = 1,2, j = 1,2, \text{ for some contants } a_i^j, b_i^j.$$

Denote

$$g(r,\theta) = r^N[[(a\cos(N\theta) + b\sin(N\theta), c\cos(N\theta) + d\sin(N\theta))]]$$

$$+r^N[[(\tilde{a}\cos(N\theta) + \tilde{b}\sin(N\theta), \tilde{c}\cos(N\theta) + \tilde{d}\sin(N\theta))]], 0 < r < 1, 0 < \theta < 2\pi$$

where $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are constants.
Denote
$h_1 = r^N g_1(\theta) = (r^N(a\cos(N\theta) + b\sin(N\theta)), r^N(c\cos(N\theta) + d\sin(N\theta))) = (f_1, f_2)$.
In the polar coordinate,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\cdot\cos\theta + \frac{\partial}{\partial\theta}\cdot(\frac{-\sin\theta}{r}),$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\cdot\sin\theta + \frac{\partial}{\partial\theta}\cdot(\frac{\cos\theta}{r}).$$

Do the computation,

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial r}\cos\theta + \frac{\partial f_1}{\partial\theta}(\frac{-\sin\theta}{r})$$

$$= Nr^{N-1}(a\cos(N\theta) + b\sin(N\theta))\cos\theta + r^N(-aN\sin(N\theta) + bN\cos(N\theta))\frac{-\sin\theta}{r}$$
$$= Nr^{N-1}(a\cos(N\theta)\cos\theta + b\sin(N\theta)\cos\theta + a\sin(N\theta)\sin\theta - b\cos(N\theta)\sin\theta)$$
$$= Nr^{N-1}(a\cos((N-1)\theta) + b\sin((N-1)\theta)).$$

Similarly, $\frac{\partial f_2}{\partial x} = Nr^{N-1}(c\cos((N-1)\theta) + d\sin((N-1)\theta))$.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_1}{\partial r}\sin\theta + \frac{\partial f_1}{\partial\theta}(\frac{\cos\theta}{r})$$

$$= Nr^{N-1}(a\cos(N\theta) + b\sin(N\theta))\sin\theta + r^N(-aN\sin(N\theta) + bN\cos(N\theta))\frac{\cos\theta}{r}$$
$$= Nr^{N-1}(a\cos(N\theta)\sin\theta + b\sin(N\theta)\sin\theta - a\sin(N\theta)\cos\theta + b\cos(N\theta)\cos\theta)$$
$$= Nr^{N-1}(b\cos((N-1)\theta) - a\sin((N-1)\theta))$$

Similarly, $\frac{\partial f_2}{\partial y} = Nr^{N-1}(d\cos((N-1)\theta) - c\sin((N-1)\theta))$.
Let $(N-1)\theta = \phi$,

$$\frac{\partial h_1}{\partial x} = (\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}) = Nr^{N-1}(a\cos\phi + b\sin\phi, c\cos\phi + d\sin\phi),$$
$$\frac{\partial h_1}{\partial y} = (\frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial y}) = Nr^{N-1}(b\cos\phi - a\sin\phi, d\cos\phi - c\sin\phi).$$

$|\frac{\partial h_1}{\partial x}|^2 = N^2 r^{2(N-1)}[a^2\cos^2\phi + b^2\sin^2\phi + 2ab\sin\phi\cos\phi + c^2\cos^2\phi + d^2\sin^2\phi + 2cd\sin\phi\cos\phi]$
$|\frac{\partial h_1}{\partial y}|^2 = N^2 r^{2(N-1)}[b^2\cos^2\phi + a\sin^2\phi - 2ab\sin\phi\cos\phi + d^2\cos^2\phi + c^2\sin^2\phi - 2cd\sin\phi\cos\phi]$.
$< \frac{\partial h_1}{\partial x}, \frac{\partial f}{\partial y} >= N^2 r^{2(N-1)}[ab\cos^2\phi - a^2\sin\phi\cos\phi + b^2\sin\phi\cos\phi - ab\sin^2\phi + cd\cos^2\phi - c^2\sin\phi\cos\phi + d^2\sin\phi\cos\phi - cd\sin^2\phi]$.
Using the conformal condition, and after simplification, we have

$$(a^2+c^2-b^2-d^2)\cos^2\phi + (b^2+d^2-a^2-c^2)\sin^2\phi + (4ab+4cd)\sin\phi\cos\phi = 0,$$

and

$$(ab+cd)\cos^2\phi - (ab+cd)\sin^2\phi + (b^2+d^2-a^2-c^2)\sin\phi\cos\phi = 0.$$

While the first one can be further reduced to

$$(a^2+c^2-b^2-d^2)\cos(2\phi) + (2ab+2cd)\sin(2\phi) = 0,$$

and the second one can be reduced to

$$(ab+cd)\cos(2\phi) + \frac{b^2+d^2-a^2-c^2}{2}\sin(2\phi) = 0.$$

In a matrix form, that is equivalent to

$$\begin{pmatrix} a^2+c^2-b^2-d^2 & 2(ab+cd) \\ ab+cd & -\frac{a^2+c^2-b^2-d^2}{2} \end{pmatrix} \begin{pmatrix} \cos(2\phi) \\ \sin(2\phi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for any $\phi = (N-1)\theta$. Therefore, we must have

$$a^2+c^2-b^2-d^2 = 0, ab+cd = 0.$$

Similarly, we have

$$\tilde{a}^2+\tilde{c}^2-\tilde{b}^2-\tilde{d}^2 = 0, \tilde{a}\tilde{b}+\tilde{c}\tilde{d} = 0.$$

Now we will discuss the solutions of above equations.
If $c = 0$, then $ab = 0$, i.e $a = 0$ or $b = 0$.
If $c = a = 0$, then $b = d = 0$.
If $c = b = 0$, then $a = \pm d$.
If $d = 0$, then $ab = 0$, i.e. $a = 0$ or $b = 0$.
If $d = a = 0$, then $c = \pm b$.
If $d = b = 0$, then $a = c = 0$.
Now we assume that $cd \neq 0$, let $a = kc$, $b = ld$, for some constants k, l.

$$a^2+c^2-b^2-d^2 = k^2c^2+c^2-l^2d^2-d^2 = (k^2+1)c^2-(l^2+1)d^2 = 0$$
$$ab+cd = (kl+1)cd = 0.$$

Since $cd \neq 0$, $kl = -1$, i.e. $l = -1/k$.
So $(k^2+1)c^2 = (l^2+1)d^2 = (\frac{1}{k^2}+1)d^2 = \frac{k^2+1}{k^2}d^2$.
Hence $d^2 = k^2c^2$, i.e. $d = \pm kc$, $b = ld = -\frac{1}{k} \cdot \pm kc = \mp c$.
In a word, here are the possible solutions of a, b, c, d:

$$\begin{aligned}
&(1)\ (a,b,c,d) = (d,0,0,d), d \neq 0\\
&(2)\ (a,b,c,d) = (-d,0,0,d), d \neq 0\\
&(3)\ (a,b,c,d) = (0,b,b,0), b \neq 0\\
&(4)\ (a,b,c,d) = (0,b,-b,0), b \neq 0\\
&(5)\ (a,b,c,d) = (kc,-c,c,kc), k \neq 0, c \neq 0\\
&(6)\ (a,b,c,d) = (kc,c,c,-kc), k \neq 0, c \neq 0\\
&(7)\ (a,b,c,d) = (0,0,0,0)
\end{aligned}$$

We have the same conclusions about $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.
Finally, we will check the requirement that $\eta \circ g = \frac{1}{2}(r^N g_1(\theta) + r^N g_2(\theta))$ is Dir minimizing, i.e $r^N((a+\tilde{a})\cos(N\theta) + (b+\tilde{b})\sin(N\theta), (c+\tilde{c})\cos(N\theta) + (d+\tilde{d})\sin(N\theta))$ is Dir minimizing. Therefore, the 4-tuple $(a+\tilde{a}, b+\tilde{b}, c+\tilde{c}, d+\tilde{d})$ must be in one of the seven forms above.
Now let us consider the possibility of matching up the two 4-tuples (a,b,c,d), and $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$.
(1)+(1),i.e $(a,b,c,d) = (d,0,0,d), d \neq 0$, $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (\tilde{d},0,0,\tilde{d}), \tilde{d} \neq 0$.

$$(a+\tilde{a}, b+\tilde{b}, c+\tilde{c}, d+\tilde{d}) = (d+\tilde{d}, 0, 0, d+\tilde{d})$$

$$g(r,0) = [[r^N(d,0)]] + [[r^N(\tilde{d},0)]]$$

$$g(r,2\pi) = [[r^N(d\cos(2\pi N), d\sin(2\pi N))]] + [[r^N(\tilde{d}\cos(2\pi N), \tilde{d}\sin(2\pi N))]]$$

Let $\psi = 2\pi N$. Since g is Hölder continuous in $\mathbb{U}_1^m(0)$, $g(r,0) = g(r,2\pi)$, i.e.,

$$d\sin(2\pi N) = \tilde{d}\sin(2\pi N) = 0$$

$$\sin(2\pi N) = 0$$

$$2\pi N = k\pi, \text{i.e } N = k/2, \text{for some integer } k = 1,2,\cdots$$

Case 1:

$$d\cos(2\pi N) = d, \tilde{d}\cos(2\pi N) = \tilde{d}$$

$$\cos(2\pi N) = 1, \text{i.e } 2\pi N = 2k\pi, \text{for some integer } k = 1,2,\cdots$$

$$N = k, \text{for some integer } k = 1,2,\cdots$$

Therefore, $g(r,\theta) = dr^k[[(\cos(k\theta), \sin(k\theta))]] + \tilde{d}r^k[[(\cos(k\theta), \sin(k\theta))]]$.
Case 2:

$$d = \tilde{d}\cos(2\pi N), \tilde{d} = d\cos(2\pi N)$$

$$\cos(2\pi N) = \pm 1, \text{i.e.} 2\pi N = k\pi$$

$$N = k/2, \text{for some integer } k = 1, 2, \cdots$$

If N is an integer, then $d = \tilde{d}$.

$$g(r, \theta) = 2dr^k[[(\cos(k\theta), \sin(k\theta))]],$$

in which case $\Sigma(g) = \emptyset$, a contradiction to our assumption. Hence $N = k/2$, for some odd integer k

$$d = -\tilde{d}$$

$$g(r, \theta) = dr^{k/2}[[(\cos(\theta k/2), \sin(\theta k/2))]] + (-d)r^{k/2}[[(\cos(\theta k/2), \sin(\theta k/2))]]$$

(1)+(2): $(a, b, c, d) = (d, 0, 0, d), d \neq 0, (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (-\tilde{d}, 0, 0, \tilde{d}), \tilde{d} \neq 0$.

$$(a, b, c, d) + (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (d - \tilde{d}, 0, 0, d + \tilde{d})$$

which is in neither of those seven forms.
(1)+(3): $(a, b, c, d) = (d, 0, 0, d), d \neq 0, (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (0, \tilde{b}, \tilde{b}, 0), \tilde{d} \neq 0$

$$(a, b, c, d) + (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (d, \tilde{b}, \tilde{b}, d)$$

which is in neither of those seven forms.
(1)+(4): $(a, b, c, d) = (d, 0, 0, d), d \neq 0, (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (0, \tilde{b}, -\tilde{b}, 0), \tilde{d} \neq 0$.

$$(a, b, c, d) + (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (d, \tilde{b}, -\tilde{b}, d)$$

which is of form (5).

$$g(r, 0) = [[r^N(d, 0)]] + [[r^N(0, -\tilde{b})]]$$

$$g(r, 2\pi) = [[r^N(d\cos(2\pi N), d\sin(2\pi N))]] + [[r^N(\tilde{b}\sin(2\pi N), -\tilde{b}\cos(2\pi N))]]$$

Case 1:

$$d = d\cos(2\pi N), 0 = d\sin(2\pi N), 0 = \tilde{b}\sin(2\pi N), -\tilde{b} = -\tilde{b}\cos(2\pi N)$$

$$\cos(2\pi N) = 1, \sin(2\pi N) = 0$$

$$N = k, \text{for some integer } k = 1, 2, \cdots$$

$$g(r, \theta) = dr^k[[(\cos(k\theta), \sin(k\theta))]] + \tilde{b}r^k[[(\sin(k\theta), -\cos(k\theta))]]$$

Case 2:

$$d = \tilde{b}\sin(2\pi N), 0 = -\tilde{b}\cos(2\pi N), 0 = d\cos(2\pi N), -\tilde{b} = d\sin(2\pi N)$$

which has no solutions.
(1)+(5). $(a, b, c, d) = (d, 0, 0, d), d \neq 0, (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (l\tilde{c}, -\tilde{c}, \tilde{c}, l\tilde{c}), l \neq 0, \tilde{c} \neq 0$.

$$(a, b, c, d) + (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (d + l\tilde{c}, -\tilde{c}, \tilde{c}, d + l\tilde{c})$$

$$g(r, 0) = [[r^N(d, 0)]] + [[r^N(l\tilde{c}, \tilde{c})]]$$

$$g(r, 2\pi) = [[r^N(d\cos(2\pi N), d\sin(2\pi N))]] +$$

$$[[r^N(l\tilde{c}\cos(2\pi N)-\tilde{c}\sin(2\pi N),\tilde{c}\cos(2\pi N)+l\tilde{c}\sin(2\pi N))]]$$

Case 1:

$$d=l\tilde{c}\cos\psi-\tilde{c}\sin\psi, 0=\tilde{c}\cos\psi+l\tilde{c}\sin\psi$$

$$l\tilde{c}=d\cos\psi, \tilde{c}=d\sin\psi$$

No solution.
Case 2:

$$d=d\cos\psi, 0=d\sin\psi,$$

$$l\tilde{c}=l\tilde{c}\cos\psi-\tilde{c}\sin\psi, \tilde{c}=\tilde{c}\cos\psi+l\tilde{c}\sin\psi$$

The solution is $\cos\psi=1, \sin\psi=0$, hence $2\pi N=\psi=2k\pi,\ N=k$.

$$g(r,\theta)=dr^k[[(\cos(k\theta),\sin(k\theta))]]+\tilde{c}r^k[[(l\cos(k\theta)-\sin(k\theta),\cos(k\theta)+l\sin(k\theta))]]$$

(1)+(6). $(a,b,c,d)=(d,0,0,d), d\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(l\tilde{c},\tilde{c},\tilde{c},-l\tilde{c}), l\neq 0, \tilde{c}\neq 0$.

$$(a+\tilde{a},b+\tilde{b},c+\tilde{c},d+\tilde{d})=(d+l\tilde{c},\tilde{c},\tilde{c},d-l\tilde{c})$$

which is in neither of the seven forms above.
(1)+(7). $(a,b,c,d)=(d,0,0,d), d\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(0,0,0,0)$

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(d,0,0,d)$$

$$g(r,0)=r^N[[(d,0)]]+r^N[[(0,0)]]$$

$$g(r,2\pi)=r^N[[(d\cos(2\pi N),d\sin(2\pi N))]]+r^N[[(0,0)]]$$

Since $d\neq 0$, the only possible matching up is

$$d=d\cos\psi, 0=d\sin\psi$$

i.e $\sin\psi=0, \cos\psi=1$.

$$N=k,\ \text{for some positivie integer } k=1,2,\cdots$$

$$g(r,\theta)=dr^k[[(\cos(k\theta),\sin(k\theta))]]+[[(0,0)]]$$

(2)+(2). $(a,b,c,d)=(-d,0,0,d), d\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(-\tilde{d},0,0,\tilde{d}), \tilde{d}\neq 0$.

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(-d-\tilde{d},0,0,d+\tilde{d})$$

$$g(r,0)=[[r^N(-d,0)]]+[[r^N(-\tilde{d},0)]]$$

$$g(r,2\pi)=[[r^N(-d\cos\psi,d\sin\psi)]]+[[r^N(-\tilde{d}\cos\psi,\tilde{d}\sin\psi)]]$$

Case 1:

$$-d=-d\cos\psi, 0=d\sin\psi, -\tilde{d}=-\tilde{d}\cos\psi, 0=\tilde{d}\sin\psi$$

Hence $\cos\psi=1, \sin\psi=0$, i.e.,

$$N=k,\ \text{for some positive integer } k=1,2,\cdots.$$

$$g(r,\theta) = dr^k[[(-\cos(k\theta), \sin(k\theta))]] + \tilde{d}r^k[[(-\cos(k\theta), \sin(k\theta))]]$$

Case 2:

$$-d = -\tilde{d}\cos\psi, 0 = \tilde{d}\sin\psi, -\tilde{d} = -d\cos\psi, 0 = d\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = \pm 1$.
If $\cos\psi = 1$, then $d = \tilde{d}$, which means that $\Sigma(g) = \emptyset$, a contradiction to our assumption. Therefore $\cos\psi = -1$, which means $N = k/2$ for some odd integer k. Moreover, we get $d = -\tilde{d}$.

$$g(r,\theta) = \pm dr^k[[(-\cos(k\theta), \sin(k\theta))]]$$

(2)+(3). $(a,b,c,d) = (-d,0,0,d), d \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,\tilde{b},\tilde{b},0), \tilde{b} \neq 0$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (-d,\tilde{b},\tilde{b},d)$$

$$g(r,0) = [[r^N(-d,0)]] + [[r^N(0,\tilde{b})]]$$

$$g(r,2\pi) = [[r^N(-d\cos\psi, d\sin\psi)]] + [[r^N(\tilde{b}\sin\psi, \tilde{b}\cos\psi)]]$$

Case 1:

$$-d = -d\cos\psi, 0 = d\sin\psi, 0 = \tilde{b}\sin\psi, \tilde{b} = \tilde{b}\cos\psi$$

Hence $\sin\psi = 0, \cos\psi = 1$, i.e. $N = k$, for some positive integer $k = 1, 2, \cdots$.

$$g(r,\theta) = d[[r^k(-\cos(k\theta), \sin(k\theta))]] + \tilde{b}r^k[[(\sin(k\theta), \cos(k\theta))]]$$

Case 2:

$$-d = \tilde{b}\sin\psi, 0 = \tilde{b}\cos\psi, 0 = -d\cos\psi, \tilde{b} = d\sin\psi$$

No solution.
(2)+(4) $(a,b,c,d) = (-d,0,0,d), d \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,\tilde{b},-\tilde{b},0), \tilde{b} \neq 0$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (-d,\tilde{b},-\tilde{b},d)$$

which is in neither of the seven forms above.
(2)+(5) $(a,b,c,d) = (-d,0,0,d), d \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c},-\tilde{c},\tilde{c},l\tilde{c}), l \neq 0, \tilde{c} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}-d,-\tilde{c},\tilde{c},l\tilde{c}+d)$$

which is in neither of the seven forms above.
(2)+(6) $(a,b,c,d) = (-d,0,0,d), d \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c},\tilde{c},\tilde{c},-l\tilde{c}), l \neq 0, \tilde{c} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}-d,\tilde{c},\tilde{c},d-l\tilde{c})$$

$$g(r,0) = r^N[[(-d,0)]] + r^N[[(l\tilde{c},\tilde{c})]]$$

$$g(r,2\pi) = r^N[[(-d\cos\psi, d\sin\psi)]] + r^N[[(l\tilde{c}\cos\psi + \tilde{c}\sin\psi, \tilde{c}\cos\psi - l\tilde{c}\sin\psi)]]$$

Case 1:

$$-d = -d\cos\psi, 0 = d\sin\psi$$

$$l\tilde{c} = l\tilde{c}\cos\psi + \tilde{c}\sin\psi, \tilde{c} = \tilde{c}\cos\psi - l\tilde{c}\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = 1$, i.e. $N = k$, for some potitive integer k.

$g(r,\theta) = dr^k[[(-\cos(k\theta), \sin(k\theta))]] + \tilde{c}r^k[[(l\cos(k\theta)+\sin(k\theta), \cos(k\theta)-l\sin(k\theta))]]$

Case 2:

$$-d = l\tilde{c}\cos\psi + \tilde{c}\sin\psi, 0 = \tilde{c}\cos\psi - l\tilde{c}\sin\psi$$

$$l\tilde{c} = -d\cos\psi, \tilde{c} = d\sin\psi$$

No solution.
(2)+(7). $(a,b,c,d) = (-d,0,0,d), d \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,0,0,0)$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (-d,0,0,d)$$

$$g(r,0) = r^N[[(-d,0)]] + r^N[[(0,0)]]$$

$$g(r,2\pi) = r^N[[(-d\cos\psi, d\sin\psi)]] + r^N[[(0,0)]]$$

Since $d \neq 0$, there is only one way of matching up:

$$-d = -d\cos\psi, 0 = d\sin\psi$$

Therefore $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(-d\cos(k\theta), d\sin(k\theta))]] + [[(0,0)]]$$

(3)+(3): $(a,b,c,d) = (0,b,b,0), b \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,\tilde{b},\tilde{b},0), \tilde{b} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0, b+\tilde{b}, b+\tilde{b}, 0)$$

$$g(r,0) = r^N[[(0,b)]] + r^N[[(0,\tilde{b})]]$$

$$g(r,2\pi) = r^N[[(b\sin\psi, b\cos\psi)]] + r^N[[(\tilde{b}\sin\psi, \tilde{b}\cos\psi)]]$$

Case 1:

$$0 = b\sin\psi, b = b\cos\psi, 0 = \tilde{b}\sin\psi, \tilde{b} = \tilde{b}\cos\psi$$

The solution is $\sin\psi = 0, \cos\psi = 1$, i.e. $N = k$ for some positive integer k.

$$g(r,\theta) = br^k[[(\sin(k\theta), \cos(k\theta))]] + \tilde{b}r^k[[(\sin(k\theta), \cos(k\theta))]]$$

Case 2:

$$0 = \tilde{b}\sin\psi, b = \tilde{b}\cos\psi, 0 = b\sin\psi, \tilde{b} = b\cos\psi$$

The solution is $\sin\psi = 0, \cos\psi = \pm 1$.
If $\cos\psi = 1$, then $b = \tilde{b}$, which means $\Sigma(g) = \emptyset$, a contradiction to our assumption.Therefore $\cos\psi = -1$,$b = -\tilde{b}$ and $N = k/2$ for some odd integer k.

$$g(r,\theta) = br^{k/2}[[(\sin(\theta k/2), \cos(\theta k/2))]] + (-b)r^{k/2}[[(\sin(\theta k/2), \cos(\theta k/2))]]$$

(3)+(4) $(a,b,c,d) = (0,b,b,0), b \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,\tilde{b},-\tilde{b},0), \tilde{b} \neq 0$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0, b+\tilde{b}, b-\tilde{b}, 0)$$

which is in neither of the seven forms above.
(3)+(5) $(a,b,c,d)=(0,b,b,0), b\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(l\tilde{c},-\tilde{c},\tilde{c},l\tilde{c}), l\neq 0, \tilde{c}\neq 0$.

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(l\tilde{c},b-\tilde{c},b+\tilde{c},l\tilde{c})$$

which is in neither of the seven forms above.
(3)+(6) $(a,b,c,d)=(0,b,b,0), b\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(l\tilde{c},\tilde{c},\tilde{c},-l\tilde{c}), l\neq 0, \tilde{c}\neq 0$.

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(l\tilde{c},b+\tilde{c},b+\tilde{c},-l\tilde{c})$$

$$g(r,0)=r^N[[(0,b)]]+r^N[[(l\tilde{c},\tilde{c})]]$$

$$g(r,2\pi)=r^N[[(b\sin\psi,b\cos\psi)]]+r^N[[(l\tilde{c}\cos\psi+\tilde{c}\sin\psi,\tilde{c}\cos\psi-l\tilde{c}\sin\psi)]]$$

Case 1:

$$0=b\sin\psi, b=b\cos\psi$$

$$l\tilde{c}=l\tilde{c}\cos\psi+\tilde{c}\sin\psi, \tilde{c}=\tilde{c}\cos\psi-l\tilde{c}\sin\psi$$

The solution is $\sin\psi=0,\cos\psi=1$, hence $N=k$ for some positive integer k.

$$g(r,\theta)=br^k[[(\sin(k\theta),\cos(k\theta))]]+r^k[[(l\tilde{c}\cos(k\theta)+\tilde{c}\sin(k\theta),\tilde{c}\cos(k\theta)-l\tilde{c}\sin(k\theta))]]$$

Case 2:

$$0=l\tilde{c}\cos\psi+\tilde{c}\sin\psi, b=\tilde{c}\cos\psi-l\tilde{c}\sin\psi$$

$$l\tilde{c}=b\sin\psi, \tilde{c}=b\cos\psi$$

No solution.
(3)+(7) $(a,b,c,d)=(0,b,b,0), b\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(0,0,0,0)$.

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(0,b,b,0)$$

$$g(r,0)=r^N[[(0,b)]]+[[(0,0)]]$$

$$g(r,2\pi)=r^N[[(b\sin\psi,b\cos\psi)]]+[[(0,0)]]$$

Since $b\neq 0$, there is only one way of matching up:

$$0=b\sin\psi, b=b\cos\psi$$

Hence $\sin\psi=0,\cos\psi=1$, i.e. $N=k$ for some positive integer k.

$$g(r,\theta)=br^k[[(\sin(k\theta),\cos(k\theta))]]+[[(0,0)]]$$

(4)+(4) $(a,b,c,d)=(0,b,-b,0), b\neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(0,\tilde{b},-\tilde{b},0), \tilde{b}\neq 0$.

$$(a,b,c,d)+(\tilde{a},\tilde{b},\tilde{c},\tilde{d})=(0,b+\tilde{b},-(b+\tilde{b}),0)$$

$$g(r,0)=r^N[[(0,-b)]]+r^N[[(0,-\tilde{b})]]$$

$$g(r,2\pi)=r^N[[(b\sin\psi,-b\cos\psi)]]+r^N[[(\tilde{b}\sin\psi,-\tilde{b}\cos\psi)]]$$

Case 1:

$$0 = b\sin\psi, -b = -b\cos\psi, 0 = \tilde{b}\sin\psi, -\tilde{b} = -\tilde{b}\cos\psi$$

The solution is $N = k$ for some positive integer k.

$$g(r,\theta) = br^k[[(\sin(k\theta), -\cos(k\theta))]] + \tilde{b}r^k[[(\sin(k\theta), -\cos(k\theta))]]$$

Case 2:

$$0 = \tilde{b}\sin\psi, -b = -\tilde{b}\cos\psi, 0 = b\sin\psi, -\tilde{b} = -b\cos\psi$$

The solution is $\sin\psi = 0, \cos\psi = \pm 1$.
If $\cos\psi = 1$, then $b = \tilde{b}$, which means $\Sigma(g) = \emptyset$, a contradiction to our assumption. Therefore $\cos\psi = -1$, hence $b = -\tilde{b}$, and $N = k/2$ for some odd integer k.

$$g(r,\theta) = br^{k/2}[[(\sin(\theta k/2), -\cos(\theta k/2))]] + (-b)r^{k/2}[[(\sin(\theta k/2), -\cos(\theta k/2))]]$$

(4)+(5) $(a,b,c,d) = (0,b,-b,0), b \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}, -\tilde{c}, \tilde{c}, l\tilde{c}), l \neq 0, \tilde{c} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}, b - \tilde{c}, -b + \tilde{c}, l\tilde{c})$$

$$g(r,0) = r^N[[(0,-b)]] + r^N[[(l\tilde{c},\tilde{c})]]$$

$$g(r,2\pi) = r^N[[(b\sin\psi, -b\cos\psi)]] + r^N[[(l\tilde{c}\cos\psi - \tilde{c}\sin\psi, \tilde{c}\cos\psi + l\tilde{c}\sin\psi)]]$$

Case 1

$$0 = b\sin\psi, -b = -b\cos\psi$$

$$l\tilde{c} = l\tilde{c}\cos\psi - \tilde{c}\sin\psi, \tilde{c} = \tilde{c}\cos\psi + l\tilde{c}\sin\psi$$

The solution is $N = k$ for some positive integer k.

$$g(r,\theta) = br^k[[(\sin(k\theta), -\cos(k\theta))]] + \tilde{c}r^k[[(l\cos(k\theta) - \sin(k\theta), \cos(k\theta) + l\sin(k\theta))]]$$

Case 2

$$0 = l\tilde{c}\cos\psi - \tilde{c}\sin\psi, -b = \tilde{c}\cos\psi + l\tilde{c}\sin\psi$$

$$l\tilde{c} = b\sin\psi, \tilde{c} = -b\cos\psi$$

No solutions.
(4)+(6) $(a,b,c,d) = (0,b,-b,0), b \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}, \tilde{c}, \tilde{c}, -l\tilde{c}), l \neq 0, \tilde{c} \neq 0$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (l\tilde{c}, b + \tilde{c}, -b + \tilde{c}, -l\tilde{c})$$

which is in neither of the seven forms above.
(4)+(7) $(a,b,c,d) = (0,b,-b,0), b \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,0,0,0)$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,b,-b,0)$$

$$g(r,0) = r^N[[(0,-b)]] + [[(0,0)]]$$

$$g(r,2\pi) = r^N[[(b\sin\psi, -b\cos\psi)]] + [[(0,0)]]$$

Since $b \neq 0$, there is only one way of matching up

$$0 = b\sin\psi, -b = -b\cos\psi$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(b\sin(k\theta), -b\cos(k\theta))]] + [[(0,0)]]$$

(5)+(5) $(a,b,c,d) = (lc, -c, c, lc), l \neq 0, c \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (\tilde{l}\tilde{c}, -\tilde{c}, \tilde{c}, \tilde{l}\tilde{c}), \tilde{l} \neq 0, \tilde{c} \neq 0$

$$(a+\tilde{a}, b+\tilde{b}, c+\tilde{c}, d+\tilde{d}) = (lc+\tilde{l}\tilde{c}, -c-\tilde{c}, c+\tilde{c}, lc+\tilde{l}\tilde{c})$$

$$g(r,0) = r^N[[(lc,c)]] + r^N[[(\tilde{l}\tilde{c},\tilde{c})]]$$

$$g(r,2\pi) = r^N[[(lc\cos\psi - c\sin\psi, c\cos\psi + lc\sin\psi)]] + r^N[[(\tilde{l}\tilde{c}\cos\psi - \tilde{c}\sin\psi, \tilde{c}\cos\psi + \tilde{l}\tilde{c}\sin\psi)]]$$

Case 1

$$lc = lc\cos\psi - c\sin\psi, c = c\cos\psi + lc\sin\psi$$

$$\tilde{l}\tilde{c} = \tilde{l}\tilde{c}\cos\psi - \tilde{c}\sin\psi, \tilde{c} = \tilde{c}\cos\psi + \tilde{l}\tilde{c}\sin\psi$$

The solution is $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(lc\cos(k\theta) - c\sin(k\theta), c\cos(k\theta) + lc\sin(k\theta))]]$$

$$+r^k[[(\tilde{l}\tilde{c}\cos(k\theta) - \tilde{c}\sin(k\theta), \tilde{c}\cos(k\theta) + \tilde{l}\tilde{c}\sin(k\theta))]]$$

Case 2

$$lc = \tilde{l}\tilde{c}\cos\psi - \tilde{c}\sin\psi, c = \tilde{c}\cos\psi + \tilde{l}\tilde{c}\sin\psi$$

$$\tilde{l}\tilde{c} = lc\cos\psi - c\sin\psi, \tilde{c} = c\cos\psi + lc\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = \pm 1$.
If $\cos\psi = 1$, then $l = \tilde{l}$ and $c = \tilde{c}$, which means $\Sigma(g) = \emptyset$, a contradiction to our assumption. Therefore $\cos\psi = -1$, hence $N = k/2$, for some odd integer k, $c = -\tilde{c}, l = \tilde{l}$

$$g(r,\theta) = \pm r^{k/2}[[(lc\cos(\theta k/2) - c\sin(\theta k/2), c\cos(\theta k/2) + lc\sin(\theta k/2))]]$$

(5)+(6) $(a,b,c,d) = (lc, -c, c, lc), l \neq 0, c \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (\tilde{l}\tilde{c}, \tilde{c}, \tilde{c}, -\tilde{l}\tilde{c}), \tilde{l} \neq 0, \tilde{c} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (lc+\tilde{l}\tilde{c}, -c+\tilde{c}, c+\tilde{c}, lc-\tilde{l}\tilde{c})$$

which is in neither of the seven forms above.
(5)+(7) $(a,b,c,d) = (lc, -c, c, lc), l \neq 0, c \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,0,0,0)$.

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (lc, -c, c, lc)$$

$$g(r,0) = r^N[[(lc,c)]] + [[(0,0)]]$$

$$g(r,2\pi) = r^N[[(lc\cos\psi - c\sin\psi, c\cos\psi + lc\sin\psi)]] + [[(0,0)]]$$

Since $c \neq 0$, there is only one way of matching up

$$lc = lc\cos\psi - c\sin\psi, c = c\cos\psi + lc\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = 1$, hence $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(lc\cos(k\theta) - c\sin(k\theta), c\cos(k\theta) + lc\sin(k\theta))]] + [[(0,0)]]$$

(6)+(6) $(a,b,c,d) = (lc,c,c,-lc), l \neq 0, c \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{c}) = (\tilde{l}\tilde{c},\tilde{c},\tilde{c},-\tilde{l}\tilde{c}), \tilde{l} \neq 0, \tilde{c} \neq 0$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (lc+\tilde{l}\tilde{c}, c+\tilde{c}, c+\tilde{c}, -lc-\tilde{l}\tilde{c})$$

$$g(r,0) = r^N[[(lc,c)]] + r^N[[(\tilde{l}\tilde{c},\tilde{c})]]$$

$$g(r,2\pi) = r^N[[(lc\cos\psi + c\sin\psi, c\cos\psi - lc\sin\psi)]] + r^N[[(\tilde{l}\tilde{c}\cos\psi + \tilde{c}\sin\psi, \tilde{c}\cos\psi - \tilde{l}\tilde{c}\sin\psi)]]$$

Case 1:

$$lc = lc\cos\psi + c\sin\psi, c = c\cos\psi - lc\sin\psi$$

$$\tilde{l}\tilde{c} = \tilde{l}\tilde{c}\cos\psi + \tilde{c}\sin\psi, \tilde{c} = \tilde{c}\cos\psi - \tilde{l}\tilde{c}\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = 1$, i.e $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(lc\cos(k\theta) + c\sin(k\theta), c\cos(k\theta) - lc\sin(k\theta))]]$$

$$+r^k[[(\tilde{l}\tilde{c}\cos(k\theta) + \tilde{c}\sin(k\theta), \tilde{c}\cos(k\theta) - \tilde{l}\tilde{c}\sin(k\theta))]]$$

Case 2:

$$lc = \tilde{l}\tilde{c}\cos\psi + \tilde{c}\sin\psi, c = \tilde{c}\cos\psi - \tilde{l}\tilde{c}\sin\psi$$

$$\tilde{l}\tilde{c} = lc\cos\psi + c\sin\psi, \tilde{c} = c\cos\psi - lc\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = \pm 1$.
If $\cos\psi = 1$, then $c = \tilde{c}, l = \tilde{l}$, which means $\Sigma(g) = \emptyset$, a contradiction to our assumption. Therefore $\cos\psi = -1$, hence $N = k/2$ for some odd integer k and $c = -\tilde{c}, l = \tilde{l}$.

$$g(r,\theta) = \pm r^{k/2}[[(lc\cos(\theta k/2) + c\sin(\theta k/2), c\cos(\theta k/2) - lc\sin(\theta k/2))]]$$

(6)+(7) $(a,b,c,d) = (lc,c,c,-lc), l \neq 0, c \neq 0, (\tilde{a},\tilde{b},\tilde{c},\tilde{d}) = (0,0,0,0)$

$$(a,b,c,d) + (\tilde{a},\tilde{b},\tilde{c},\tilde{c}) = (lc,c,c,-lc)$$

$$g(r,0) = r^N[[(lc,c)]] + [[(0,0)]]$$

$$g(r,2\pi) = r^N[[(lc\cos\psi + c\sin\psi, c\cos\psi - lc\sin\psi)]] + [[(0,0)]]$$

Since $c \neq 0$, the only matching up is

$$lc = lc\cos\psi + c\sin\psi, c = c\cos\psi - lc\sin\psi$$

The solution is $\sin\psi = 0, \cos\psi = 1$, hence $N = k$ for some positive integer k.

$$g(r,\theta) = r^k[[(lc\cos(k\theta) + c\sin(k\theta), c\cos(k\theta) - lc\sin(k\theta))]] + [[(0,0)]]$$

(7)+(7) This case does not happen because otherwise $g \equiv 2[[0]]$.

3.2 $\Sigma = \emptyset$

If $\Sigma = \emptyset$, applying Theorem 2.3(2) to the function g, more specifically, by choosing $W = \mathbb{U}_1^2(0)$, we get

$$g(x) = 2[[g_1(x)]], x \in \mathbb{U}_1^2(0),$$

for some minimizing harmonic function $g_1 : \mathbb{U}_1^2(0) \to \mathbb{R}^2$.
The same argument as above tells that

$$g_1(r,\theta) = (r^N(a\cos(N\theta) + b\sin(N\theta)), r^N(c\cos(N\theta) + d\sin(N\theta)))$$

for some constants a, b, c, d in one of the six forms (1)-(6) above.
Case (1). $(a, b, c, d) = (d, 0, 0, d)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (dr^N, 0) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(d\cos\psi), r^N(d\sin\psi)) \end{aligned}$$

Therefore, $d = d\cos\psi, 0 = d\sin\psi$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k.$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(dr^k\cos(k\theta), dr^k\sin(k\theta))]],$$

for some nonzero constant d and some positive integer k.
Case (2). $(a, b, c, d) = (-d, 0, 0, d)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (-dr^N, 0) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(-d\cos\psi), r^N(d\sin\psi)) \end{aligned}$$

Therefore $-d = -d\cos\psi, 0 = d\sin\psi$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(-dr^k\cos(k\theta), dr^k\sin(k\theta))]],$$

for some nonzero constant d and some positive integer k.
Case (3). $(a, b, c, d) = (0, b, b, 0)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (0, br^N) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(b\sin\psi), r^N(b\cos\psi)) \end{aligned}$$

Therefore $0 = b\sin\psi, b = b\cos\psi$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(br^k\sin(k\theta), br^k\cos(k\theta))]],$$

for some nonzero constant b and some positive integer k.
Case (4). $(a,b,c,d) = (0,b,-b,0)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (0, -br^N) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(b\sin\psi), r^N(-b\cos\psi)) \end{aligned}$$

Therefore $0 = b\sin\psi, -b = -b\cos\psi$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(br^k\sin(k\theta), -br^k\cos(k\theta))]],$$

for some nonzero constant b and some positive integer k.
Case (5). $(a,b,c,d) = (lc,-c,c,lc)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (lcr^N, cr^N) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(lc\cos\psi - c\sin\psi), r^N(c\cos\psi + lc\sin\psi)) \end{aligned}$$

Therefore $lc = lc\cos\psi - c\sin\psi, c = c\cos\psi + lc\sin\psi$. Solving that gives $\cos\psi = 1, \sin\psi = 0$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(r^k(lc\cos(k\theta) - c\sin(k\theta)), r^k(c\cos(k\theta) + lc\sin(l\theta)))]],$$

for some nonzero constant l, c and some positive integer k.
Case (6). $(a,b,c,d) = (lc,c,c,-lc)$.

$$\begin{aligned} g_1(r,0) &= (ar^N, cr^N) = (lcr^N, cr^N) \\ g_1(r,2\pi) &= (r^N(a\cos\psi + b\sin\psi), r^N(c\cos\psi + d\sin\psi)) \\ &= (r^N(lc\cos\psi + c\sin\psi), r^N(c\cos\psi - lc\sin\psi)) \end{aligned}$$

Therefore $lc = lc\cos\psi + c\sin\psi, c = c\cos\psi - lc\sin\psi$. Solving that gives $\cos\psi = 1, \sin\psi = 0$, i.e.,

$$2\pi N = 2k\pi, \text{ for some positive integer } k$$

Hence $N = k$ for some positive integer k.

$$g(r,\theta) = 2[[(r^k(lc\cos(k\theta) + c\sin(k\theta)), r^k(c\cos(k\theta) - lc\sin(k\theta)))]],$$

for some nonzero constant l, c and some positive integer k.

$\square$

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